Polynomial Reproduction of Multivariate Scalar Subdivision Schemes with General Dilation

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Abstract

In this paper we study scalar multivariate subdivision schemes with general integer expanding dilation matrix. Our main result yields simple algebraic conditions on the symbols of such schemes that characterize their polynomial reproduction, i.e. their capability to generate exactly the same polynomials from which the initial data is sampled. These algebraic conditions also allow us to determine the approximation order of the associated refinable functions and to choose the "correct" parametrization, i.e. the grid points to which the newly computed values are attached at each subdivision iteration. We use this special choice of the parametrization to increase the degree of polynomial reproduction of known subdivision schemes and to construct new schemes with given degree of polynomial reproduction.

Keywords: multivariate scalar subdivision schemes, expanding dilation matrix, polynomial reproduction, refinement parametrization

1. Introduction

In [2], the authors derive simple algebraic conditions on the subdivision symbol of a multivariate scalar subdivision scheme with dilation matrix $M = mI$, $m \in \mathbb{Z}$, $|m| > 1$, that allow us to determine the degree of

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polynomial reproduction of such a scheme and, consequently by [16], its approximation order. In this paper we extend the results of [2] to the case of general expanding dilation matrix $M \in \mathbb{Z}^{s\times s}$ whose spectral radius $\rho(M)$ satisfies $\rho(M) > 1$. On the one hand, our interest in the study of such a general dilation matrix M is motivated by the existence of multivariate scalar subdivision schemes for which the results of [2] are not applicable, e.g.

$$M = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \quad a, b, c \in \mathbb{Z}, \quad |a|, |c| > 1,$$

see [10, 14]. On the other hand, even for schemes whose dilation matrix M satisfies $M^\ell = mI$ for some integer $\ell > 1$ and $m \in \mathbb{Z}$, $|m| > 1$, e.g., $\sqrt{3}-$subdivision with $M^2 = -3I$, the results of this paper allow us to directly investigate the properties of the associated symbol, instead of dealing with its $\ell-$th iterated version.

Moreover, in contrast to polynomial generation, i.e. the capability of a scheme to generate the full space of polynomials of certain degree, there are very few theoretical results on polynomial reproduction of subdivision schemes, (see [4, 9, 13]). Thus, we strongly believe that our paper may further contribute to the development of this topic. See [2] for a detailed discussion on polynomial generation and reproduction of subdivision schemes.

In addition, from a practical point of view, the results in this paper, if combined with those in [3], provide a simple algebraic tool for the construction of new interesting multivariate subdivision schemes with enhanced properties. Note that, in the univariate case, affine combinations of existing schemes have been already used in [5–8, 11–13] for such purpose.

The remainder of this paper is organized as follows. In section 2, we briefly summarize key notions concerning subdivision schemes with integer, expanding dilation matrix M. In section 3, we prove that for a non-singular multivariate scalar subdivision scheme with finitely supported mask $\mathrm{a} = \{\mathrm{a}_{\boldsymbol{\alpha}},\ \boldsymbol{\alpha} \in \mathbb{Z}^s\}$ and symbol $a(\boldsymbol{z}) = \sum_{\boldsymbol{\alpha}\in\mathbb{Z}^s} \mathrm{a}_{\boldsymbol{\alpha}}\, \boldsymbol{z}^{\boldsymbol{\alpha}}$, the polynomial reproduction of degree up to k is equivalent to

$$\big(D^{\mathbf{j}}a\big)(1,\ldots,1) = |\det(M)| \prod_{i=1}^{s} \prod_{\ell_i=0}^{j_i-1} (\tau_i - \ell_i) \quad \text{and} \quad \big(D^{\mathbf{j}}a\big)(\boldsymbol{\varepsilon}) = 0 \tag{1.1}$$

for $\boldsymbol{\varepsilon} \in \Xi'$, $\mathbf{j} \in \mathbb{N}_0^s$ and $|\mathbf{j}| \le k$. The set Ξ' in (1.1) is a finite set of certain multi-indices and $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_s) \in \mathbb{R}^s$ appears in the parametrization associated with the subdivision scheme. The importance of condition (1.1) for

$k = 1$ is that it allows us to identify the correct parametrization of any non-singular or even only convergent subdivision scheme to guarantee at least the reproduction of linear polynomials. The parametrization determines the grid points to which the newly computed values are attached at each step of the subdivision recursion to ensure the highest degree of polynomial reproduction of the scheme. Since some preliminary results discussed in [2] are still valid for a general integer expanding matrix M, in the following we only prove the extensions of [2, Propositions 2.3 and 2.5]. In section 4, we use the results in section 3 to increase the degree of polynomial reproduction of existing subdivision schemes by considering their affine combinations.

2. Background and notation

Let $M \in \mathbb{Z}^{s\times s}$ be an expanding *dilation matrix* whose spectral radius $\rho(M)$ satisfies $\rho(M) > 1$. The $r-$th step of a scalar s-variate subdivision scheme is given by

$$\mathrm{d}^{(r+1)}_{\boldsymbol{\alpha}} = (\mathcal{S}_{\mathrm{a}}\mathrm{d}^{(r)})_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} \mathrm{a}_{\boldsymbol{\alpha}-M\boldsymbol{\beta}}\mathrm{d}^{(r)}_{\boldsymbol{\beta}}\ , \quad \mathrm{d}^{(0)} \in \ell(\mathbb{Z}^s), \quad r \geq 0\ , \quad \boldsymbol{\alpha} \in \mathbb{Z}^s, \tag{2.1}$$

and is defined by the finitely supported *mask* $\mathrm{a} = \{\mathrm{a}_{\boldsymbol{\alpha}} \in \mathbb{R},\ \boldsymbol{\alpha} \in \mathbb{Z}^s\}$ and the *subdivision operator*

$$\mathcal{S}_{\mathrm{a}} : \ell(\mathbb{Z}^s) \to \ell(\mathbb{Z}^s), \quad \big(\mathcal{S}_{\mathrm{a}}\mathrm{d}\big)_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} \mathrm{a}_{\boldsymbol{\alpha}-M\boldsymbol{\beta}}\, \mathrm{d}_{\boldsymbol{\beta}}\ , \quad \boldsymbol{\alpha} \in \mathbb{Z}^s\ , \tag{2.2}$$

where $\ell(\mathbb{Z}^s)$ is the space of scalar sequences indexed by $\mathbb{Z}^s$. We denote the *symbol* of a scalar s-variate subdivision scheme with mask a by

$$a(\boldsymbol{z}) = \sum_{\boldsymbol{\alpha}\in\mathbb{Z}^s} \mathrm{a}_{\boldsymbol{\alpha}}\, \boldsymbol{z}^{\boldsymbol{\alpha}}, \quad \boldsymbol{z} = (z_1, \ldots, z_s) \in (\mathbb{C}\setminus\{0\})^s\ , \quad \boldsymbol{z}^{\boldsymbol{\alpha}} = z_1^{\alpha_1} z_2^{\alpha_2} \cdot \ldots \cdot z_s^{\alpha_s}.$$

Let $m := |\det(M)|$ and

$$\mathrm{E} = \{\boldsymbol{e}_0, \ldots, \boldsymbol{e}_{m-1}\} \tag{2.3}$$

be the set of representatives of $\mathbb{Z}^s/M\mathbb{Z}^s$ containing $\mathbf{0} = (0, 0, \ldots, 0)$. The m *submasks* of a and the associated symbols $a_{\boldsymbol{e}}(\boldsymbol{z})$, $\boldsymbol{e} \in E$, are defined by $\{\mathrm{a}_{M\boldsymbol{\alpha}+\boldsymbol{e}},\ \boldsymbol{\alpha} \in \mathbb{Z}^s\}$ and $a_{\boldsymbol{e}}(\boldsymbol{z}) = \sum_{\boldsymbol{\alpha}\in\mathbb{Z}^s} \mathrm{a}_{M\boldsymbol{\alpha}+\boldsymbol{e}}\, \boldsymbol{z}^{M\boldsymbol{\alpha}+\boldsymbol{e}}$, respectively.

Definition 2.1 (Condition Z_k). *We say that the mask symbol $a(\boldsymbol{z})$ satisfies the* zero condition of order k (Condition Z_k), *if*

$$a(\mathbf{1}) = m \quad and \quad \left(D^{\mathbf{j}} a\right)(\boldsymbol{\varepsilon}) = 0 \quad for \quad \boldsymbol{\varepsilon} \in \Xi' := \Xi \setminus \{\mathbf{1}\} \quad and \quad |\mathbf{j}| < k\,, \tag{2.4}$$

where $\Xi = \{\varepsilon_0, \ldots, \varepsilon_{m-1}\}$ is the set of representatives of $\mathbb{Z}^s / M^T \mathbb{Z}^s$ which contains $\mathbf{1} = (1, 1, \ldots, 1)$.

For $\boldsymbol{\tau} \in \mathbb{R}^s$ and $\boldsymbol{\alpha} \in \mathbb{Z}^s$ define

$$\mathbf{t}_{\boldsymbol{\alpha}}^{(r)} := \mathbf{t}_{\mathbf{0}}^{(r)} + M^{-r}\boldsymbol{\alpha}, \quad \mathbf{t}_{\mathbf{0}}^{(r)} = \mathbf{t}_{\mathbf{0}}^{(r-1)} - M^{-r}\boldsymbol{\tau}, \quad \mathbf{t}_{\mathbf{0}}^{(0)} = 0, \quad r \geq 0. \tag{2.5}$$

We call the sequence $\{\mathbf{t}^{(r)},\ r \geq 0\}$, $\mathbf{t}^{(r)} = \{\mathbf{t}_{\boldsymbol{\alpha}}^{(r)},\ \alpha \in \mathbb{Z}^s\}$, the *sequence of parameter values associated with the subdivision scheme.*

Definition 2.2. *If the sequence of continuous functions $\{F^{(r)},\ r \geq 0\}$ with $F^{(r)}(\mathbf{t}_{\boldsymbol{\alpha}}^{(r)}) = \mathrm{d}_{\boldsymbol{\alpha}}^{(r)}$, $\boldsymbol{\alpha} \in \mathbb{Z}^s$, converges for all initial data $\mathrm{d}^{(0)} = \{\mathrm{d}_{\boldsymbol{\alpha}}^{(0)},\ \boldsymbol{\alpha} \in \mathbb{Z}^s\} \in \ell(\mathbb{Z}^s)$, then we denote its limit by $S_{\mathrm{a}}^{\infty}\mathrm{d}^{(0)} = \lim_{r\to\infty} F^{(r)}$ and say that S_{a} is convergent.*

Definition 2.3. *A subdivision scheme is called* non-singular, *if it is convergent, and $S_{\mathrm{a}}^{\infty}\mathrm{d}^{(0)} = 0$ if and only if $\mathrm{d}_{\boldsymbol{\alpha}}^{(0)} = 0$ for all $\boldsymbol{\alpha} \in \mathbb{Z}^s$.*

We denote by Π_k the space of multivariate polynomials of total degree $k \geq 0$.

Definition 2.4. *A convergent subdivision scheme S_{a} with parametrization $\{\mathbf{t}^{(r)},\ r \geq 0\}$ is* reproducing *polynomials up to degree γ_R if for any $\pi \in \Pi_{\gamma_R}$ and $\mathrm{d}^{(0)} = \{\pi(\mathbf{t}_{\boldsymbol{\alpha}}^{(0)}),\ \boldsymbol{\alpha} \in \mathbb{Z}^s\}$ the limit of the subdivision process satisfies $S_{\mathrm{a}}^{\infty}\mathrm{d}^{(0)} = \pi$.*

3. Algebraic conditions for polynomial reproduction

In this section we extend the results of Proposition 2.3 and Proposition 2.5 in [2] to the case of a general expanding dilation matrix $M \in \mathbb{Z}^{s\times s}$.

Proposition 3.1. *Let S_{a} be a non-singular s-variate scalar subdivision scheme that generates linear polynomials, i.e. its symbol satisfies Condition Z_2. Then S_{a} reproduces linear polynomials if and only if its parameter values are given by (2.5) with*

$$\boldsymbol{\tau} = m^{-1}\left(D^{\boldsymbol{\epsilon}_1} a(\mathbf{1}), \ldots, D^{\boldsymbol{\epsilon}_s} a(\mathbf{1})\right),$$

where $D^{\mathbf{j}}$ is the $\mathbf{j}$-th directional derivative and $\boldsymbol{\epsilon}_\ell$ is the ℓ-th unit vector of $\mathbb{R}^s$.

Proof: By [2, Proposition 1.7] it suffices to prove the claim for the step-wise polynomial reproduction. Moreover, any convergent subdivision scheme reproduces constants, hence we only consider the starting sequences $\mathrm{d}_{\boldsymbol{\alpha}}^{(r)} = \pi(\mathbf{t}_{\boldsymbol{\alpha}}^{(r)}) = (\mathbf{t}_{\boldsymbol{\alpha}}^{(r)})_j$, $\boldsymbol{\alpha} \in \mathbb{Z}^s$ and $j = 1, \ldots, s$. Then for any $\boldsymbol{\alpha} \in \mathbb{Z}^s$ and $\boldsymbol{e} = (\boldsymbol{e}_1, \ldots, \boldsymbol{e}_s) \in E$ we get

$$
\begin{aligned}
\mathrm{d}_{M\boldsymbol{\alpha}+\boldsymbol{e}}^{(r+1)} &= \sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} \mathrm{a}_{M(\boldsymbol{\alpha}-\boldsymbol{\beta})+\boldsymbol{e}}\, \mathrm{d}_{\boldsymbol{\beta}}^{(r)} = \sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} \mathrm{a}_{M\boldsymbol{\beta}+\boldsymbol{e}}\, \mathrm{d}_{\boldsymbol{\alpha}-\boldsymbol{\beta}}^{(r)} \\
&= \sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} \mathrm{a}_{M\boldsymbol{\beta}+\boldsymbol{e}} \left(\mathbf{t}_{\mathbf{0}}^{(r)} + M^{-r}(\boldsymbol{\alpha}-\boldsymbol{\beta})\right)_j \\
&= \sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} \mathrm{a}_{M\boldsymbol{\beta}+\boldsymbol{e}} \left(\mathbf{t}_{\mathbf{0}}^{(r)} + M^{-(r+1)}(M\boldsymbol{\alpha}+\boldsymbol{e})\right)_j - \sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} \mathrm{a}_{M\boldsymbol{\beta}+\boldsymbol{e}} \left(M^{-(r+1)}(M\boldsymbol{\beta}+\boldsymbol{e})\right)_j \\
&= \left(\mathbf{t}_{\mathbf{0}}^{(r)} + M^{-(r+1)}(M\boldsymbol{\alpha}+\boldsymbol{e})\right)_j - \left(M^{-(r+1)} \sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} \mathrm{a}_{M\boldsymbol{\beta}+\boldsymbol{e}}(M\boldsymbol{\beta}+\boldsymbol{e})\right)_j \\
&= \left(\mathbf{t}_{\mathbf{0}}^{(r)} + M^{-(r+1)}(M\boldsymbol{\alpha}+\boldsymbol{e})\right)_j - \left(M^{-(r+1)}(D^{\boldsymbol{\epsilon}_1} a_{\boldsymbol{e}}(\mathbf{1}), \ldots, D^{\boldsymbol{\epsilon}_s} a_{\boldsymbol{e}}(\mathbf{1}))\right)_j \\
&= \left(\mathbf{t}_{\mathbf{0}}^{(r)} + M^{-(r+1)}(M\boldsymbol{\alpha}+\boldsymbol{e}) - m^{-1}M^{-(r+1)}(D^{\boldsymbol{\epsilon}_1} a(\mathbf{1}), \ldots, D^{\boldsymbol{\epsilon}_s} a(\mathbf{1}))\right)_j,
\end{aligned}
$$

where we use the fact that $\sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} \mathrm{a}_{M\boldsymbol{\beta}+\boldsymbol{e}} = a_{\boldsymbol{e}}(\mathbf{1}) = 1$ and the last equality is due to [2, Proposition 2.1, part (ii)] for $\mathbf{j} = \boldsymbol{\epsilon}_j$. Thus, $\mathrm{d}_{M\boldsymbol{\alpha}+\boldsymbol{e}}^{(r+1)}$ is equal to

$$
\pi(\mathbf{t}_{M\boldsymbol{\alpha}+\boldsymbol{e}}^{(r+1)}) = t_{\mathbf{0},j}^{(r+1)} + \left(M^{-(r+1)}(M\boldsymbol{\alpha}+\boldsymbol{e})\right)_j = t_{\mathbf{0},j}^{(r)} + \left(M^{-(r+1)}(M\boldsymbol{\alpha}-\boldsymbol{\tau}+\boldsymbol{e})\right)_j,
$$

$\alpha \in \mathbb{Z}^s$, if and only if $\tau_j = m^{-1} D^{\boldsymbol{\epsilon}_j} a(\mathbf{1})$ for all $j = 1, \cdots, s$. ■

As in [2] we easily get the following consequence of Proposition 3.1 for convergent subdivision schemes.

Corollary 3.2. *Let S_{a} be a convergent s-variate scalar subdivision scheme that generates linear polynomials, i.e. its symbol satisfies Condition Z_2. Then S_{a} reproduces linear polynomials if its parameter values are given by (2.5) with*

$$
\boldsymbol{\tau} = m^{-1} \left(D^{\boldsymbol{\epsilon}_1} a(\mathbf{1}), \ldots, D^{\boldsymbol{\epsilon}_s} a(\mathbf{1})\right).
$$

The next Proposition is crucial for the proof of our main result.

Proposition 3.3. *Let $k \in \mathbb{N}$, $\boldsymbol{\tau} \in \mathbb{R}^s$ and $q_{\mathbf{j}}$ given by*

$$q_{\mathbf{0}}(\boldsymbol{z}) := 1, \quad q_{\mathbf{j}}(z_1, \ldots, z_s) := \prod_{i=1}^{s} \prod_{\ell_i=0}^{j_i-1} (z_i - \ell_i), \quad \mathbf{j} = (j_1, \ldots, j_s), \quad \boldsymbol{z} \in \mathbb{R}^s. \tag{3.1}$$

A subdivision symbol $a(\boldsymbol{z})$ satisfies

$$\big(D^{\mathbf{j}} a\big)(\mathbf{1}) = m \; q_{\mathbf{j}}(\boldsymbol{\tau}), \quad \big(D^{\mathbf{j}} a\big)(\boldsymbol{\varepsilon}) = 0 \quad \boldsymbol{\varepsilon} \in \Xi', \quad \mathbf{j} \in \mathbb{N}_0^s, \quad |\mathbf{j}| \le k, \tag{3.2}$$

if and only if

$$\sum_{\boldsymbol{\beta} \in \mathbb{Z}^s} \mathrm{a}_{\boldsymbol{\alpha} - M\boldsymbol{\beta}} \left(M^{-r} \boldsymbol{\beta}\right)^{\mathbf{j}} = \left(M^{-(r+1)}(\boldsymbol{\alpha} - \boldsymbol{\tau})\right)^{\mathbf{j}}, \; \alpha \in \mathbb{Z}^s, \; \mathbf{j} \in \mathbb{N}_0^s, \; |\mathbf{j}| \le k, \; r \ge 0\,. \tag{3.3}$$

Proof: Note that due to [2, Proposition 2.1- part (*iii*)] conditions in (3.2) are equivalent to

$$q_{\mathbf{j}}(\boldsymbol{\tau}) = \sum_{\boldsymbol{\beta} \in \mathbb{Z}^s} q_{\mathbf{j}}(\boldsymbol{\alpha} - M\boldsymbol{\beta}) \mathrm{a}_{\boldsymbol{\alpha} - M\boldsymbol{\beta}}, \quad \mathbf{j} \in \mathbb{N}_0^s, \quad |\mathbf{j}| \le k, \quad \boldsymbol{\alpha} \in \mathbb{Z}^s\,. \tag{3.4}$$

The proof is by induction on k. For $k = 0$ we get, for any $\boldsymbol{\tau} \in \mathbb{R}^s$, $q_{\mathbf{0}}(\boldsymbol{\tau}) = \sum_{\boldsymbol{\beta} \in \mathbb{Z}^s} \mathrm{a}_{\boldsymbol{\alpha} - M\boldsymbol{\beta}} = \left(M^{-(r+1)}(\boldsymbol{\alpha} - \boldsymbol{\tau})\right)^{\mathbf{0}} = 1$. Assume next that the claim is true for all $\mathbf{j} \in \mathbb{N}_0^s$ with $|\mathbf{j}| \le k-1$ and prove it for $\mathbf{j} \in \mathbb{N}_0^s$ with $|\mathbf{j}| = k$. The polynomial $q_{\mathbf{j}}$ in $\boldsymbol{x}$ of (total) degree $|\mathbf{j}| = k$ is of the form

$$q_{\mathbf{j}}(\boldsymbol{\alpha} - M^{r+1}\boldsymbol{x}) = \sum_{\boldsymbol{\ell} \in \mathbb{N}_0^s, \; |\boldsymbol{\ell}| \le k} c_{\mathbf{j}, \boldsymbol{\alpha}, \boldsymbol{\ell}} \boldsymbol{x}^{\boldsymbol{\ell}}, \quad \boldsymbol{x} \in \mathbb{R}^s\,. \tag{3.5}$$

Therefore, using the induction assumption and by (3.4) and (3.5) we have

$$
\begin{aligned}
q_{\mathbf{j}}(\boldsymbol{\tau}) &= \sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} q_{\mathbf{j}}(\boldsymbol{\alpha} - M^{r+1}M^{-r}\boldsymbol{\beta})\mathrm{a}_{\boldsymbol{\alpha}-M\boldsymbol{\beta}} = \sum_{\boldsymbol{\beta}\in\mathbb{Z}^s}\ \sum_{\boldsymbol{\ell}\in\mathbb{N}_0^s,\ |\boldsymbol{\ell}|\le k} c_{\mathbf{j},\boldsymbol{\alpha},\boldsymbol{\ell}}\ \mathrm{a}_{\boldsymbol{\alpha}-M\boldsymbol{\beta}} \left(M^{-r}\boldsymbol{\beta}\right)^{\boldsymbol{\ell}} \\
&= \sum_{\boldsymbol{\ell}\in\mathbb{N}_0^s,\ |\boldsymbol{\ell}|=k} c_{\mathbf{j},\boldsymbol{\alpha},\boldsymbol{\ell}} \sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} \mathrm{a}_{\boldsymbol{\alpha}-M\boldsymbol{\beta}} \left(M^{-r}\boldsymbol{\beta}\right)^{\boldsymbol{\ell}} + \sum_{\boldsymbol{\ell}\in\mathbb{N}_0^s,\ |\boldsymbol{\ell}|\le k-1} c_{\mathbf{j},\boldsymbol{\alpha},\boldsymbol{\ell}} \sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} \mathrm{a}_{\boldsymbol{\alpha}-M\boldsymbol{\beta}} \left(M^{-r}\boldsymbol{\beta}\right)^{\boldsymbol{\ell}} \\
&= \sum_{\boldsymbol{\ell}\in\mathbb{N}_0^s,\ |\boldsymbol{\ell}|=k} c_{\mathbf{j},\boldsymbol{\alpha},\boldsymbol{\ell}} \sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} \mathrm{a}_{\boldsymbol{\alpha}-M\boldsymbol{\beta}} \left(M^{-r}\boldsymbol{\beta}\right)^{\boldsymbol{\ell}} + \sum_{\boldsymbol{\ell}\in\mathbb{N}_0^s,\ |\boldsymbol{\ell}|\le k-1} c_{\mathbf{j},\boldsymbol{\alpha},\boldsymbol{\ell}} \left(M^{-(r+1)}(\boldsymbol{\alpha}-\boldsymbol{\tau})\right)^{\boldsymbol{\ell}} \\
&= \sum_{\boldsymbol{\ell}\in\mathbb{N}_0^s,\ |\boldsymbol{\ell}|=k} c_{\mathbf{j},\boldsymbol{\alpha},\boldsymbol{\ell}} \left(\sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} \mathrm{a}_{\boldsymbol{\alpha}-M\boldsymbol{\beta}} \left(M^{-r}\boldsymbol{\beta}\right)^{\boldsymbol{\ell}} - \left(M^{-(r+1)}(\boldsymbol{\alpha}-\boldsymbol{\tau})\right)^{\boldsymbol{\ell}} \right) + q_{\mathbf{j}}(\boldsymbol{\tau})\,.
\end{aligned}
$$

The last equality is due to the fact that

$$
q_{\mathbf{j}}(\boldsymbol{\tau}) = q_{\mathbf{j}}\left(\boldsymbol{\alpha} - M^{r+1}M^{-(r+1)}(\boldsymbol{\alpha}-\boldsymbol{\tau})\right) = \sum_{\boldsymbol{\ell}\in\mathbb{N}_0^s,\ |\boldsymbol{\ell}|\le k} c_{\mathbf{j},\boldsymbol{\alpha},\boldsymbol{\ell}} \left(M^{-(r+1)}(\boldsymbol{\alpha}-\boldsymbol{\tau})\right)^{\boldsymbol{\ell}}.
$$

Note that all rows and columns of M^{r+1}, $r \ge 0$, are non-zero vectors, due to $\rho(M) > 1$ and $det(M^{r+1}) = det(M)^{r+1}$. Thus, if all diagonal elements of M^{r+1} are non-zero, $c_{\mathbf{j},\boldsymbol{\alpha},\mathbf{j}} \ne 0$. If one of the diagonal elements of M^{r+1} is zero, then there exists an $s \times s$ permutation matrix P, such that for any $\mathbf{j} \in \mathbb{N}_0^s$, $|\mathbf{j}| = k$, the coefficient $c_{\mathbf{j},\boldsymbol{\alpha},P\mathbf{j}} \ne 0$ in

$$
\sum_{\boldsymbol{\ell}\in\mathbb{N}_0^s,\ |\boldsymbol{\ell}|=k} c_{\mathbf{j},\boldsymbol{\alpha},\boldsymbol{\ell}} \left(\sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} \mathrm{a}_{\boldsymbol{\alpha}-M\boldsymbol{\beta}} \left(M^{-r}\boldsymbol{\beta}\right)^{\boldsymbol{\ell}} - \left(M^{-(r+1)}(\boldsymbol{\alpha}-\boldsymbol{\tau})\right)^{\boldsymbol{\ell}} \right) = 0.
$$

Hence, comparing the coefficients on both sides of the above identity, we note that it is satisfied for all $\mathbf{j} \in \mathbb{N}_0^s$, $|\mathbf{j}| = k$, if and only if

$$
\sum_{\boldsymbol{\beta}\in\mathbb{Z}^s} \mathrm{a}_{\boldsymbol{\alpha}-M\boldsymbol{\beta}} \left(M^{-r}\boldsymbol{\beta}\right)^{\mathbf{j}} - \left(M^{-(r+1)}(\boldsymbol{\alpha}-\boldsymbol{\tau})\right)^{\mathbf{j}} = 0, \quad \text{for all}\, \mathbf{j} \in \mathbb{N}_0^s, \quad |\mathbf{j}| = k. \qquad \blacksquare
$$

The following result follows directly from [2, Theorem 2.6] and Proposition 3.3 by replacing mI by $M \in \mathbb{Z}^{s\times s}$ whenever appropriate.

Theorem 3.4. *Let $k \in \mathbb{N}_0$. A non-singular s-variate scalar subdivision scheme with symbol $a(\boldsymbol{z})$ and associated parametrization in (2.5) with some $\boldsymbol{\tau} \in \mathbb{R}^s$ reproduces polynomials of degree up to k if and only if*

$$\big(D^{\mathbf{j}} a\big)(\mathbf{1}) = m\ q_{\mathbf{j}}(\boldsymbol{\tau}) \quad \textit{and} \quad \big(D^{\mathbf{j}} a\big)(\boldsymbol{\varepsilon}) = 0 \quad \textit{for} \quad \boldsymbol{\varepsilon} \in \Xi', \quad |\mathbf{j}| \le k.$$

We conclude by observing that, if a scheme is only convergent, then Theorem 3.4 leads to the following sufficient conditions for determining the degree of polynomial reproduction of the scheme.

Corollary 3.5. *Let $k \in \mathbb{N}_0$. A convergent s-variate scalar subdivision scheme with symbol $a(\boldsymbol{z})$ and associated parametrization in (2.5) with some $\boldsymbol{\tau} \in \mathbb{R}^s$ reproduces polynomials of degree up to k if*

$$\big(D^{\mathbf{j}} a\big)(\mathbf{1}) = m\ q_{\mathbf{j}}(\boldsymbol{\tau}) \quad \textit{and} \quad \big(D^{\mathbf{j}} a\big)(\boldsymbol{\varepsilon}) = 0 \quad \textit{for} \quad \boldsymbol{\varepsilon} \in \Xi', \quad |\mathbf{j}| \le k\,.$$

4. Applications and examples

4.1. Box splines

The results of [3] imply that any bivariate convergent subdivision scheme whose symbol satisfies Condition Z_3 is an affine combination of the 3−directional box spline symbols from the set

$$\{B_{j,j,h},\ B_{j,h,j},\ B_{h,j,j}\ :\ h = 0,1, \quad j = 3-h\}$$

with

$$B_{h,i,j}(\boldsymbol{z}) = 4 \cdot \left(\frac{1+z_1}{2}\right)^h \cdot \left(\frac{1+z_2}{2}\right)^i \cdot \left(\frac{1+z_1 z_2}{2}\right)^j, \quad h,i,j \in \mathbb{N}_0.$$

One easily checks that e.g. the affine combination

$$a(\boldsymbol{z}) = 5 \cdot B_{221}(\boldsymbol{z}) - B_{212}(\boldsymbol{z}) - B_{122}(\boldsymbol{z}) - 2 \cdot B_{330}(\boldsymbol{z})$$

satisfies all conditions of Corollary 3.5 for $k = 3$ although none of the summands of this affine combination does separately. The subdivision scheme associated with $a(\boldsymbol{z})$ is C^1 as its second difference operator is contractive.

4.2. $\sqrt{3}-$ subdivision

This example shows how to determine the degree of polynomial reproduction of the approximating $\sqrt{3}-$subdivision schemes given in [15, page 21] from the corresponding mask symbol $a(\boldsymbol{z})$ instead of the iterated symbol $a(z_1 z_2^{-2}, z_1^2 z_2^{-1}) \cdot a(\boldsymbol{z})$, as it is done in [2]. We also show how to use affine combinations of these schemes to improve their degree of polynomial reproduction.

The dilation matrix in this case is

$$M = \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix}, \quad M^2 = -3I,$$

and, e.g., the mask symbol is given by

$$\begin{aligned} a(\boldsymbol{z}) &= \frac{1}{6}\left(z_1 z_2 + z_1^{-1} z_2^{-1} + z_1^{-1} z_2^2 + z_1^{-2} z_2 + z_1 z_2^{-2} + z_1^2 z_2^{-1}\right) \\ &+ \frac{1}{3}\left(z_1^{-1} + z_2 + z_1 z_2^{-1}\right) + \frac{1}{3}\left(z_2^{-1} + z_1 + z_1^{-1} z_2\right). \end{aligned}$$

The associated subdivision scheme satisfies zero conditions of at most order 2, see [15]. The result of Corollary 3.2 yields $\boldsymbol{\tau} = (0, 0)$, which implies that the corresponding scheme reproduces linear polynomials, if we set $\boldsymbol{\tau} = (0, 0)$ in (2.5). Since, the mask symbol satisfies at most zero conditions of order 2, the associated refinable function has approximation order 2, see [16]. Note that, similarly, the corresponding $\boldsymbol{\tau}$ is $(0, 0)$ for all approximating $\sqrt{3}-$subdivision schemes given in [15, page 21].

Take next the symbols $a_j(\boldsymbol{z})$, $j = 1, 2, 3, 4$ of the four approximating subdivision schemes in [15, page 21] that satisfy zero conditions of order 3 and consider their affine combination

$$a(\boldsymbol{z}) = \sum_{j=1}^{4} \lambda_j \cdot a_j(\boldsymbol{z}), \quad \sum_{j=1}^{4} \lambda_j = 1.$$

The resulting scheme still satisfies Condition Z_3 and in addition the rest of the conditions in Corollary 3.5 for $k = 3$, if

$$\lambda_1 = \frac{1}{2}\lambda_3 + \lambda_4 + 2, \quad \lambda_2 = -\frac{3}{2}\lambda_3 - 2\lambda_4 - 1, \quad \lambda_3, \lambda_4 \in \mathbb{R}.$$

Surprisingly, for any such choice of the parameters $\lambda_1, \ldots, \lambda_4$ the resulting scheme is given by

$$a(\boldsymbol{z}) = 1 - \frac{1}{9}(z_1^{-2} + z_1^{-2} z_2^2 + z_2^2 + z_2^2 + z_1^2 + z_1^2 z_2^{-2} + z_2^{-2}) + \frac{4}{9}(z_1^{-1} + z_1^{-1} z_2 + z_2 + z_1 + z_1 z_2^{-1} + z_2^{-1})$$

and defines the interpolatory scheme considered in [15, page 17]. Thus, by [16], this scheme has approximation order 3.

4.3. General expanding dilation

We use the results of [14] to define a convergent subdivision scheme with dilation matrix

$$M = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}. \tag{4.1}$$

Similarly, we could also treat the case of any general integer expanding dilation matrix. As shown in [14], one particular choice for the coset representatives of any such M is obtained by determining the intersection of $M \cdot [0,1)^2$ with $\mathbb{Z}^2$. Thus, for M in (4.1) we get $\mathrm{E} = \{(0,0),(1,0),(1,1),(1,2)\}$. Applying the strategy described in [1, 10] we get via convolution for the mask symbol

$$a(\boldsymbol{z}) = \sum_{\boldsymbol{e}\in\mathrm{E}} \boldsymbol{z}^{\boldsymbol{e}}$$

a convergent scheme associated with the symbol $\frac{1}{4}a^2(\boldsymbol{z})$. The symbol $a^2(\boldsymbol{z})$ satisfies (2.4) with $k=1$ and $\Xi = \{(1,1),(1,-1),(-1,\mathrm{i}),(-1,-\mathrm{i})\}$, $\mathrm{i} = \sqrt{-1}$. The conditions in (2.4) are not satisfied for $k=2$, e.g. for $\mathbf{j} = (0,2)$ and $\varepsilon = (1,-1)$. Using the result of Corollary 3.2 we get $\boldsymbol{\tau} = (2,1)$, which together with the zero conditions of order 2, by [16], implies that the scheme has approximation order 2.

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